Introduction to stochastic dispersive dynamics

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Goal: Study well-posedness of stochastic dispersive PDEs.

- stochastic nonlinear Schrödinger equation (SNLS)

$$i\partial_t u = \Delta u \pm |u|^{p-1}u + \phi\xi \qquad \mathbb{T}^d \times \mathbb{R}_+$$

spacetime white noise — bdd op on L^2 (HS: $L^2 \to H^s$)

- stochastic nonlinear wave equation (SNLW)

$$\partial_t^2 u = \Delta u \pm u^p + \xi \qquad \mathbb{T}^d \times \mathbb{R}_+$$

Sec 1 Mild formulation & stochastic convolution

$\xi(x,t) =$ (Gaussian) space time white noise on $\mathbb{T}^d \times \mathbb{R}_+$

$$\mathbb{E}\left[\xi(x_1,t_1)\overline{\xi(x_2,t_2)}\right] = \delta(x_1 - x_2)\delta(t_1 - t_2)$$

SNLS: Ito formulation

$$i\,du = \left(\Delta u \pm |u|^{p-1}u\right)dt + \phi\, dW$$

"$dW = \xi$"

$W(t) = L^2$- cylindrical Wiener process (BM)

$$= \sum_{n\in\mathbb{Z}^d} \beta_n(t)\, e_n(x)$$

$\{\beta_n\}$ = mutually indep $\mathbb{C}$-valued Brownian motion ②

$$\beta_n = \underbrace{\mathrm{Re}\,\beta_n + i\,\mathrm{Im}\,\beta_n}$$

indep $\mathbb{R}$-valued BM.

(For SNLW ($\mathbb{R}$-valued), we impose

$$\beta_{-n} = \overline{\beta_n}$$
)

For fixed $t > 0$.

$$\mathbb{E}\|W(t)\|_{H^s}^2 = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s}\, \mathbb{E}\big[|\beta_n(t)|^2\big]$$

$$\sim \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} < \infty$$

iff $s < -\frac{d}{2}$

- Recalling the regularity of BM (in t)

$$W \in C^{\alpha}\big(\mathbb{R}_+ ; H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)\big), \quad \alpha < 1/2$$

almost surely

· Mild formulation (= Duhamel formulation)

$$u(t) = S(t)\,u_0 \mp i \int_0^t S(t-t')\,|u|^{p-1}u(t')\,dt'$$

$$\boxed{\underbrace{-i \int_0^t S(t-t')\,\phi\, dW(t')}_{\substack{= \text{Stochastic convolution} \\ \Psi(t)}}}$$

$S(t) = e^{-it\Delta}$

In the following, we assume that

$$\phi \in HS(L^2; H^s)$$ Hilbert-Schmidt

(Recall $\phi \in HS(X;Y)$
iff $\left(\sum \|\phi e_n\|_Y^2\right)^{1/2} < \infty$, $\{e_n\}$, ONB of X)

Moreover, we assume that ϕ is homog in x.

i.e. $\phi e_n = \hat{\phi}_n e_n$

$\Longrightarrow \|\phi\|_{HS(L^2;H^s)} = \|\phi\|_{H^s}$

(Banach setting: replace HS by γ-radonifying operator.)

$$\Rightarrow \Psi(t) = \sum_n e_n \hat{\Phi}_n \underbrace{\int_0^t e^{i(t-t')n^2}\, d\beta_n(t')}_{\text{Wiener integral}}$$

Wiener integral ($\mathbb{R}$-valued case)

Given $f \in L^2([a,b])$, consider

$$I(f) = \int_a^b f \, dB$$

Then, $I(f)$ is a $\underbrace{\text{mean } 0}_{\mathbb{E}[I(f)]=0}$ Gaussian r.v.

with variance

$$\mathbb{E}\left[(I(f))^2\right] = \|f\|^2_{L^2([a,b])}$$

Rmk: ① if f is C^1, we can define it as a Paley-Wiener-Zygmund integral

$$I(f) = \int_a^b f\, dB = -\int_a^b f'(t) B(t) dt$$

$$(\text{if } f(a) = f(b) = 0)$$

② if B is $\mathbb{C}$-valued,

$$\mathbb{E}\left[(I(f))^2\right] = 2\|f\|^2_{L^2([a,b])}$$

- **Kolmogorov continuity criterion**

$\{X_t\}$ with values in a metric space

Suppose
$$\mathbb{E}\left[d(X_s, X_t)^p\right] \leq C_0 |t-s|^{1+\alpha}$$
for some $p, \alpha > 0$

Then
$$P\left(\sup_{s \neq t} \frac{d(X_s, X_t)}{|t-s|^{\alpha/p - \varepsilon}} \geq \lambda\right) \leq \frac{C_1}{\lambda^p}$$
$\forall\ 0 < \varepsilon < \alpha/p$

i.e. X_t is a.s. $\left(\frac{\alpha}{p} - \varepsilon\right)$-Hölder continuous

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ex: $B(t_2) - B(t_1) \sim N(0, t_2 - t_1)$ $\quad t_2 \geq t_1$

i.e. $\mathbb{E}\left(|B(t_2) - B(t_1)|^2\right] = t_2 - t_1$

$$\Rightarrow \mathbb{E}\left[|B(t_2) - B(t_1)|^p\right] \sim |t_2 - t_1|^{p/2}$$
$p^{p/2}$

$$\frac{\alpha}{p} = \frac{\frac{p}{2} - 1}{p} = \frac{1}{2} - \frac{1}{p} \to \frac{1}{2} \quad \text{as } p \to \infty$$

i.e. BM is a.s. $\left(\frac{1}{2} - \varepsilon\right)$-Hölder.

comes from hypercontractivity of OU semigroup

- **Wiener chaos estimate:**

$\{g_n\}$, seq of indep Gaussian r.v.'s

Given $k \in \mathbb{N}$, let $\{P_j\}_{j\in\mathbb{N}}$ be a seq of polynomials in $\bar{g} = \{g_n\}$ of deg at most k. Then, for $p \geq 2$,

$$\Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^p(\Omega)} \leq (p-1)^{k/2} \Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^2(\Omega)}$$

Prop 1: Let $\phi \in HS(L^2; H^s)$

Then, $\Psi \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}$ a.s. (slight loss of regularity)
$r \leq \infty$

$r=2$: $\Psi \in C_t H_x^s$, a.s.

Pf: $t \leq \tau$

$$\mathbb{E}\left[\Psi(x,t)\overline{\Psi(y,\tau)}\right]$$

$$= 2\sum_n e_n(x-y)|\hat{\phi}_n|^2 \underbrace{\int_0^t e^{i(t-t')n^2} e^{-i(\tau-t')n^2} dt'}$$

$$= \int_0^t e^{i(t-\tau)n^2} dt'$$

$$= t e^{i(t-\tau)n^2}$$

$\Leftarrow$ Apply $\langle \nabla_x \rangle^s$ and $\langle \nabla_y \rangle^s$

$$\Rightarrow \mathbb{E}\left[\langle \nabla_x \rangle^s \Psi(x,t) \overline{\langle \nabla_y \rangle^s \Psi(y,\tau)}\right]$$

$$= 2 \sum_n e_n(x-y) \langle n \rangle^{2s} |\hat{\Phi}_n|^2 \, t \, e^{i(t-\tau)n^2}$$

$$\Rightarrow \mathbb{E}\left[|\langle \nabla \rangle^s \Psi(x,t)|^p\right] \le p^{p/2} \, \mathbb{E}\left[|\langle \nabla \rangle^s \Psi(x,t)|^2\right]^{p/2}$$

$$\lesssim p^{p/2} t^{p/2} \|\Phi\|^p_{HS(L^2; H^s)}$$

$$\underset{p \ge r}{\Rightarrow} \Big\| \|\Psi(t)\|_{W^{s,r}_x} \Big\|_{L^p(\Omega)} \le \Big\| \|\langle \nabla \rangle^s \Psi(x,t)\|_{L^p(\Omega)} \Big\|_{L^r_x}$$

$$\lesssim p^{1/2} t^{1/2} \|\Phi\|_{HS}$$

$$\underset{\forall r < \infty}{\Rightarrow} \quad \Psi(t) \in W^{s,r}_x, \quad a.s$$

Rmk ① $r = \infty$: By Sobolev embedding thm,

$$\Big\| \|\Psi(t)\|_{W^{s,\infty}_x} \Big\|_{L^p(\Omega)} \lesssim \Big\| \|\Psi(t)\|_{W^{s+\varepsilon, r}_x} \Big\|_{L^p(\Omega)}$$

$$\lesssim p^{1/2} t^{1/2} \|\Phi\|_{HS(L^2; H^{s+\varepsilon})}$$

In other words,

$$\Psi(t) \in W^{s-\varepsilon, \infty}_x \quad \text{if } \Phi \in HS(L^2; H^s)$$

② $$\boxed{P\left(\| \Psi(t) \|_{W_x^{s,r}} > \lambda \right) \leq C e^{-c \lambda^2 / t \|\Phi\|_{HS}^2}}$$

Given $h \in \mathbb{R}$, let

$$\delta_h \Psi(x,t) = \Psi(x, t+h) - \Psi(x,t)$$

$$\Rightarrow \mathbb{E}\left[\delta_h \Psi(x,t) \cdot \overline{\delta_h \Psi(y,t)} \right]$$

$$= \mathbb{E}\left[\Psi(x,t+h) \overline{\Psi(y,t+h)} \right]$$

$$- \mathbb{E}\left[\Psi(x,t) \overline{\Psi(y,t+h)} \right]$$

$$- \mathbb{E}\left[\Psi(x,t+h) \overline{\Psi(y,t)} \right]$$

$$+ \mathbb{E}\left[\Psi(x,t) \overline{\Psi(y,t)} \right]$$

$$= 2 \sum_n e_n(x-y) |\widehat{\Phi}_n|^2 \boxed{\left\{ (t+h) - t e^{-ihn^2} - t e^{ihn^2} + t \right\}}$$

$$= F_n(t,h) \to 0 \text{ as } h \to 0$$

$$\left(\text{and } |F_n(t,h)| \leq |t| + |h| \right)$$

Note: $|F_n(t,h)| \le |h| + t\,|1-e^{-2ihn^2}|$

$$\lesssim |h| + t\min(1, |h| n^2)$$

$$\lesssim |h| + t|h|^\alpha |n|^{2\alpha}, \qquad \alpha \in [0,1]$$

$$\Rightarrow \|\langle\nabla\rangle^s \delta_h \Psi(x,t)\|_{L^p(\Omega)}$$

$$\le p^{1/2} \|\langle\nabla\rangle^s \delta_h \Psi(x,t)\|_{L^2(\Omega)}$$

$$\lesssim p^{1/2}(1+T)^{1/2}|h|^{\alpha/2}\Big(\sum_n \langle n\rangle^{2(s+\alpha)}|\hat{\Phi}_n|^2\Big)^{1/2}, \quad \forall\, t\in[0,T]$$

$p \ge r$ $\quad \| \|\delta_h \Psi(t)\|_{W_x^{s,r}} \|_{L^p(\Omega)}$

$$\lesssim C_T\, p^{1/2} \|\Phi\|_{HS(L^2; H^{s+\alpha})} |h|^{\alpha/2}, \quad \forall\, t\in[0,T]$$

Kolmogorov conti crit. (and replace $s+\alpha$ by s)

$$\Rightarrow \Psi \in C_t^{\alpha/2 - \frac{1}{p} -} W_x^{s-\alpha, r} \quad \text{a.s} \quad \text{if } \Phi \in HS(L^2; H^s)$$

$$\overset{p\to\infty}{\Longrightarrow} \Psi \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r} \quad \text{a.s} \quad \text{if } \Phi \in HS(L^2; H^s)$$

- When $r=2$: It suffices to study the continuity property of $\tilde{\Psi}(t) = S(-t)\Psi(t)$

$$\Rightarrow \mathbb{E}\big[\tilde{\Psi}(x,t)\overline{\tilde{\Psi}(y,\tau)}\big] = 2\sum_n e_n(x-y)|\hat{\Phi}_n|^2 \cdot t.$$

By repeating a similar computation,

$$\| \|\delta_h \tilde{\Psi}(t)\|_{H_x^s} \|_{L^p(\Omega)} \lesssim C_T\, p^{1/2} \|\Phi\|_{HS(L^2;H^s)} |h|^{1/2}$$

Kolmogorov conti criterion

$\Longrightarrow$ $$\tilde{\Psi} \in C_t^{\frac{1}{2}-} H_x^s, \quad a.s.$$

$\Rightarrow$ $$\Psi \in C_t H_x^s, \ a.s.$$

$$\left(\begin{aligned} &\Psi(t+h) - \Psi(t) \\ &= S(t+h)(\tilde{\Psi}(t+h) - \tilde{\Psi}(t)) \\ &+ (S(t+h) - S(t))\tilde{\Psi}(t) \end{aligned} \right.$$

$\boxed{/}$

ex: $\Phi = Id \Rightarrow \Phi \in HS(L^2; H^s)$, $s < -\frac{d}{2}$

$$\Rightarrow \Psi \in C_t^{\frac{\alpha}{2}-} W_x^{-\frac{d}{2}-\alpha-, \infty}$$

Sec 2 Pathwise local well-posedness

① SNLS : $s > \frac{d}{2}$

$$\Gamma u(t) = \Gamma_{u_0, \phi} u(t)$$
$$= S(t)u_0 \mp i \int_0^t S(t-t')|u|^{p-1}u(t')dt'$$
$$+ \Psi(t)$$

$$\cdot \ \|\Gamma u\|_{C_T H^s} \leq \|u_0\|_{H^s} + CT\|u\|_{C_T H^s}^p + \|\Psi\|_{C_T H^s}$$

$$\cdot \ \|\Gamma u_1 - \Gamma u_2\|_{C_T H^s} \leq CT\left(\|u_1\|_{C_T H^s}^{p-1} + \|u_2\|_{C_T H^s}^{p-1}\right) \times \|u_1 - u_2\|_{C_T H^s}$$

$\Rightarrow \Gamma$ is a contraction in a ball

$$B_R \subset C_T H^s$$

where $R_\omega = 2\left(\|u_0\|_{H^s} + \|\Psi\|_{C_T H^s}\right)$

provided

$$R_\omega^{p-1} T_\omega \ll 1$$

Note: $\lim_{T \to 0} \|\Psi\|_{C_T H^s} = 0$, a.s.

$$\Rightarrow \lim_{T \to 0} R^{p-1} T = 0, \quad \text{a.s}$$

Thm 2: Let $\phi \in HS(L^2; H^s)$, $s > \frac{d}{2}$.

Then, SNLS is pathwise locally well-posed in H^s

Note: local existence time $T = T_\omega$ is random

② SNLW: $d=1$, $(u, \partial_t u) = (u_0, u_1)$

$$\Gamma u(t) = \underbrace{S(t)}_{\cos(t|\nabla|)u_0 + \frac{\sin t|\nabla|}{|\nabla|}u_1}(u_0, u_1)$$

$$\pm \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} u^p(t')\,dt'$$

$$\underbrace{+ \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|}\,dW(t')}_{=\Psi(t)}$$

$$\Rightarrow \Psi \in C_t^{\frac{\alpha}{2}-} W_x^{\frac{1}{2}-\alpha-,\infty}$$

- $s = \frac{1}{2}-$. $(u_0, u_1) \in H^s \times H^{s-1}$.

$$\|\Gamma u\|_{C_T H^s} \leq \|(u_0, u_1)\|_{H^s \times H^{s-1}}$$

$$+ \underbrace{T\|u^p\|_{C_T H^{s-1}}}_{} + \|\Psi\|_{C_T H^s}$$

$$\lesssim T\|u^p\|_{C_T L^{1+}}$$

$$= T\|u\|^p_{C_T L^{p+}}$$

$$\lesssim T\|u\|^p_{C_T H^s}$$

Thm 3: 1-d SNLW with additive space time white noise is pathwise locally well-posed in $H^s \times H^{s-1}$, $s = \frac{1}{2}-$.

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③ 1-d cubic SNLS in $L^2(\mathbb{T})$

$\phi \in HS(L^2; L^2)$

Fourier restriction norm method (Bourgain '93)

$$\|u\|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau - n^2 \rangle^b \hat{u}(n,\tau) \|_{\ell^2_n L^2_\tau}$$

$$\left(\begin{array}{l} \text{lin Schrödinger } i\partial_t u = \partial_x^2 u \\ \qquad \underset{\text{FT in } x,t}{\Longrightarrow} \quad -(\tau - n^2)\hat{u}(n,\tau) = 0. \end{array} \right.$$

· local-in-time version: $T > 0$

$$\|u\|_{X^{s,b}_T} = \inf \{ \|v\|_{X^{s,b}} : v|_{[0,T]} = u \}.$$

Basic estimates: $\eta(t) = \begin{cases} 1 & \text{on } [0,1] \\ 0 & \text{on } [-\frac{1}{2}, \frac{3}{2}]^c \end{cases}$

(i) homog lin $\quad \|\eta(t) S(t) u_0\|_{X^{s,b}} \le C_b \|u_0\|_{H^s}$

$$\left(\quad \mathcal{F}_{x,t}(\eta(t) S(t) u_0) = \hat{\eta}(\tau - n^2)\hat{u}_0(n) \right.$$

(ii) nonhomog. lin : $b > 1/2$, $\theta > 0$ small

$$\left\| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') F(t')\,dt' \right\|_{X^{s,b}} \lesssim T^{\theta} \| F \|_{X^{s,b-1+\theta}}$$

(iii) L^4 - Strichartz estimate

$$\| u \|_{L^4_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \| u \|_{X^{0,3/8}}$$

(iv) $\| u \|_{C_T H^s} \lesssim \| u \|_{X^{s,b}_T}$, $\quad b > 1/2$

($\Leftarrow$ 1-d Sobolev in time)

$$\Gamma u(t) = \eta(t) S(t) u_0 \mp i\, \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t')\,dt'$$

$$+ \mathbb{1}_{[0,1]} \Psi(t)$$

Take $X^{0,\frac{1}{2}-}_T$ - norm $(T \le 1)$.

- $\| \eta(t) S(t) u_0 \|_{X^{0,\frac12-}_T} \lesssim \| u_0 \|_{L^2}$

continuity in time needs to be shown separately

$$\cdot \left\| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t')\, dt' \right\|_{X_T^{0,\frac{1}{2}-}}$$

$$\leq \| \quad \cdots \quad \|_{X_T^{0,\frac{1}{2}+}}$$

$$\leq \left\| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') |v|^2 v(t')\, dt' \right\|_{X^{0,\frac{1}{2}+}}$$

(for any extension v of u

$$\lesssim T^\theta \| |v|^2 v \|_{X^{0,-\frac{1}{2}+\theta+}}$$

$$= T^\theta \sup_{\|w\|_{X^{0,\frac{1}{2}-\theta-}}=1} \left| \iint w \cdot |v|^2 v \, dx\, dt \right|$$

$$\leq \|w\|_{L^4_{x,t}} \|v\|^3_{L^4_{x,t}}$$

$$\lesssim \|w\|_{X^{0,3/8}} \|v\|^3_{X^{0,3/8}}$$

$$\lesssim T^\theta \|v\|^3_{X^{0,3/8}}$$

Take inf over v.

$\Longrightarrow$

$$\| \mathrm{II} \|_{X_T^{0,\frac{1}{2}-}} \lesssim T^\theta \|u\|^3_{X_T^{0,3/8}}$$

Note: $\mathrm{II} \in C_t H^s$.

Let $b < 1/2$.

Prop 4: $\| \mathbb{1}_{[0,1]} \Psi \|_{X^{0,b}} < \infty$, a.s.

$$\Rightarrow \| \Gamma u \|_{X_T^{0,\frac{1}{2}-}} \lesssim \| u_0 \|_{L^2} + T^\theta \| u \|^3_{X_T^{0,\frac{1}{2}-}} + \underbrace{\| \mathbb{1}_{[0,1]} \Psi \|_{X^{0,\frac{1}{2}-}}}_{\leq C_\omega < \infty \text{ a.s.}}$$

Thm 5: Let $\Phi \in HS(L^2; L^2)$

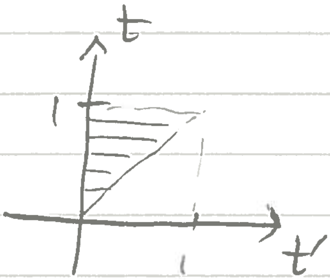
Then, 1-d cubic SNLS is pathwise locally well-posed in $L^2(\mathbb{T})$.

Proof of Prop 4: Suppose $\Phi \in HS(L^2; H^s)$

$$\Psi(t) = \sum_n e^{inx} \hat{\Phi}_n \int_0^t e^{i(t-t')n^2} d\beta_n(t')$$

$$\widehat{\mathbb{1}_{[0,1]} \Psi}(n,t) = \mathbb{1}_{[0,1]} \hat{\Phi}_n \int_0^t e^{i(t-t')n^2} d\beta_n(t')$$

$$\widehat{\mathbb{1}_{[0,1]}\Psi}(n,\tau) = \widehat{\Phi}_n \int_{\mathbb{R}} e^{-it(\tau-n^2)} \mathbb{1}_{[0,1]}(t) \int_0^t e^{-it'n^2} d\beta_n(t')\, dt$$



$$= \widehat{\Phi}_n \int_0^1 e^{-it'n^2} \underbrace{\int_{t'}^1 e^{-it(\tau-n^2)} dt}_{\lesssim \min\left(1, \frac{1}{\tau-n^2}\right) \sim \frac{1}{\langle \tau-n^2\rangle}} d\beta_n(t')$$

$$\Rightarrow \left(\mathbb{E}\, \|\mathbb{1}_{[0,1]}\Psi\|^p_{X^{s,b}}\right)^{1/p}$$

$$\lesssim \Big\| \langle n\rangle^s \langle \tau-n^2\rangle^b \underbrace{\|\widehat{\mathbb{1}_{[0,1]}\Psi}(n,\tau)\|_{L^p(\Omega)}}_{\lesssim p^{1/2} |\widehat{\Phi}_n|^2 \frac{1}{\langle \tau-n^2\rangle}} \Big\|_{\ell^2_n L^2_\tau}$$

$$\lesssim p^{1/2} \left(\sum \langle n\rangle^{2s} |\widehat{\Phi}_n|^2 \int \frac{1}{\langle \tau-n^2\rangle^{2-2b}} d\tau \right)^{1/2}$$

$$\lesssim p^{1/2} \|\Phi\|_{HS(L^2;H^s)} < \infty \quad \text{if } b < 1/2.$$

$\square$

Da Prato – Debussche trick '03 $\left\{\begin{array}{l}\text{McKean '95}\\ \text{Bourgain '96}\end{array}\right.$

④ Another look at 1-d cubic SNLS

Write $u = \Psi + v$ and consider the fixed pt problem for v.

$$v(t) = \eta(t) S(t) u_0 \mp i \eta\left(\frac{t}{T}\right) \int_0^t S(t-t') |v+\Psi|^2 (v+\Psi) dt'.$$

$\Rightarrow$ contraction in $X_T^{0,\frac{1}{2}+}$ (a bit smoother)

only change:

$$\|v+\Psi\|_{L^4_{x,T}} \le \|v\|_{L^4_{x,T}} + \underbrace{\|\Psi\|_{L^4_{x,T}}}_{<\infty,\ a.s.}$$

$\Leftarrow$ No need to take the $X^{s,b}$-norm of Ψ.

Rmk: ϕ can be random / time dependent, etc.

Sec 3 On global well-posedness

① 1-d cubic SNLS in $L^2(\mathbb{T})$

· deterministic NLS

LWP in $L^2(\mathbb{T})$ & L^2-conservation

$\Rightarrow$ By iterating LWP, we obtain GWP in $L^2(\mathbb{T})$

Idea: Control $M(u) = \sum_n |\hat{u}_n|^2 = \sum_n (p_n^2 + q_n^2)$

$$p_n = \operatorname{Re} \hat{u}_n, \quad q_n = \operatorname{Im} \hat{u}_n$$

$$d\hat{u}_n = \left(i n^2 \hat{u}_n - i \mathcal{F}_x(|u|^2 u)(n) \right) dt - i \hat{\Phi}_n \, d\beta_n$$

$$\Rightarrow \quad dp_n = \left(-n^2 q_n + \operatorname{Im} \mathcal{F}_x(|u|^2 u)(n) \right) dt + \underbrace{\operatorname{Im}(\hat{\Phi}_n \, d\beta_n)}$$

$$= \operatorname{Im} \hat{\Phi}_n \cdot d(\operatorname{Re} \beta_n) + \operatorname{Re} \hat{\Phi}_n \, d(\operatorname{Im} \beta_n)$$

$$dq_n = \left(n^2 p_n - \operatorname{Re} \mathcal{F}_x(|u|^2 u)(n) \right) dt \underbrace{- \operatorname{Re}(\hat{\Phi}_n \, d\beta_n)}$$

$$= -\operatorname{Re} \hat{\Phi}_n \, d(\operatorname{Re} \beta_n) + \operatorname{Im} \hat{\Phi}_n \, d(\operatorname{Im} \beta_n)$$

Ito's lemma : $dX = f\,dt + g\,dB$

Consider $F(X)$

Then, $dF = \partial_x F\, dX + \frac{1}{2}\partial_x^2 F\,(dX)^2$

$= \partial_x F\,(f\,dt + g\,dB)$

$+ \frac{1}{2}\partial_x^2 F \cdot g^2\,dt$

$(dt)^2 = 0$

$dt\,dB = dB\,dt = 0$

$(dB)^2 = dt$

$$\Rightarrow dM \overset{\text{Ito}}{=} 2\sum (p_n\,dp_n + q_n\,dq_n)$$

$$+ \underbrace{\sum \left((dp_n)^2 + (dq_n)^2 \right)}$$

$$= \sum \Big\{ \big(\mathrm{Im}\,\hat{\Phi}_n\, d(\mathrm{Re}\,\beta_n) + \mathrm{Re}\,\hat{\Phi}_n\, d(\mathrm{Im}\,\beta_n) \big)^2$$

$$+ \big(\mathrm{Re}\,\hat{\Phi}_n\, d(\mathrm{Re}\,\beta_n) + \mathrm{Im}\,\hat{\Phi}_n\, d(\mathrm{Im}\,\beta_n) \big)^2 \Big\}$$

$$= 2\sum |\hat{\Phi}_n|^2\,dt$$

$$= 2\,\|\Phi\|^2_{HS(L^2;L^2)}\,dt$$

$$2\sum(p_n dp_n + q_n dq_n)$$

L²-conservation

$$= 2\sum\Big\{p_n\big(-n^2 q_n + \mathrm{Im}\,\mathcal{F}_x(|u|^2u)(n)\big)dt + q_n\big(n^2 p_n - \mathrm{Re}\,\mathcal{F}_x(|u|^2u)(n)\big)dt\Big\} \quad = 0$$

$$+ 2\sum_n\Big(p_n\,\mathrm{Im}(\hat{\Phi}_n d\beta_n) - q_n\,\mathrm{Re}(\hat{\Phi}_n d\beta_n)\Big)$$

Burkholder – Davis – Gundy inequality:

X, (local) martingale

$1 \le p < \infty$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t|^p\Big] \sim \mathbb{E}\Big[\langle X_t\rangle_{[0,T]}^{p/2}\Big]$$

quadratic variation

For Ito processes $dX = f\,dt + g\,dB$

$$\langle X\rangle_{[0,t]} = \int_0^t g^2\,dt'$$

One of the terms is given by

$$2 \sum_n p_n \,\mathrm{Im}\, \hat{\Phi}_n \, d(\mathrm{Re}\,\beta_n)$$

$\Rightarrow$ Take $\mathbb{E}\Big[\sup_{t \in [0,T]} \int_0^t \cdot \, dt' \Big]$

and apply B-D-G ineq

$$\lesssim \mathbb{E}\Big[\Big(\int_0^T \underbrace{\sum_n p_n^2 |\hat{\Phi}_n|^2}_{\le \|u\|_{L^2}^2 \|\Phi\|_{L^2}^2} dt \Big)^{1/2} \Big]$$

$\left(\sum a_n b_n \le \sum a_n \sum b_n \right)$

$$\le \mathbb{E}\Big[\Big(\sup_{t \in [0,T]} M(u)(t) \Big)^{1/2} \cdot T^{1/2} \|\Phi\|_{L^2} \Big]$$

$$\le \underbrace{\varepsilon \, \mathbb{E}\Big[\sup_{t \in [0,T]} M(u)(t) \Big]}_{\text{hide in the left-hand side}} + \frac{1}{\varepsilon} T \, \mathbb{E}\Big[\|\Phi\|_{L^2}^2 \Big]$$

Putting together, we obtain

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|u(t)\|_{L^2}^2 \Big] \lesssim \|u_0\|_{L^2}^2 + C\big(T, \|\Phi\|_{HS(L^2; L^2)}\big)$$

Thm 6: 1-d cubic SNLS is globally well-posed in $L^2(\mathbb{T})$

(Similar GWP in H^1 holds for energy-subcritical SNLS in the defocusing case

Rmk: In order to justify the application of Ito's lemma, one should first consider the finite dim'l approximation

$$\begin{cases} i\, du_N = \left(\Delta u_N \pm P_{\leq N}(|u_N|^2 u_N)\right)dt + P_{\leq N}(\Phi\, dW) \\ u_N|_{t=0} = P_{\leq N} u_0 \end{cases}$$

for which (finite dim'l) Ito's lemma holds.

$\Rightarrow$ obtain an a priori bound (indep of N)

Then, proceed with an approximation argument

$u_N \to u$ (a variant of the LWP argument)

and obtain the same a priori bd for a soln u to SNLS.

- For the infinite dim'l version of Ito's lemma, see Da Prato–Zabczyk (2nd ed)

② 1-d cubic SNLW with additive space time white noise

defocusing

$$\begin{cases} \partial_t^2 u = \partial_x^2 u - u^3 + \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^1(\mathbb{T}) \end{cases}$$

With $\Psi(t) = \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} dW(t') \in C_t W_x^{\frac{1}{2}-,\infty}$

write $u = v + \Psi$. Then, the residual part v satisfies

$$\begin{cases} \partial_t^2 v = \partial_x^2 v - (v + \Psi)^3 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1) \end{cases}$$

By the local argument (exercise!!), we have

$$v \in C_{T_\omega} H^1.$$

and

$$\lim_{t \to T_{max}(\omega)} \| v(t) \|_{H^1} = \infty \quad \text{if } T_{max}(\omega) < \infty$$

<u>Idea:</u> Control $E(v) = \frac{1}{2}\int (\partial_x v)^2 + \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{4}\int v^4$

$$\partial_t E(v) = \int (\partial_t v) \Big(\underbrace{\underbrace{\partial_t^2 v - \partial_x^2 v}_{= -(v+\Psi)^3} + v^3}_{\substack{= -v^2\Psi - v\Psi^2 - \Psi^3 \\ = O(|v^2\Psi| + |\Psi|^3)}} \Big)\, dx$$

$$\overset{C-S}{\lesssim} E(v)^{1/2} \Big(\|\Psi(t)\|_{L^\infty_x} E(v)^{1/2} + \|\Psi(t)\|^3_{L^6_x} \Big)$$

$$\lesssim \|\Psi\|^6_{L^\infty_T L^6_x} + \big(1 + \|\Psi\|^2_{L^\infty_{T,x}}\big) E(v)$$

$\Rightarrow$ By Gronwall, $E(v^\omega) \le C(T, \Psi^\omega) < \infty$

Thm 7: 1-d cubic SNLW with additive space time white noise is globally well-posed in $\mathcal{H}^1(\mathbb{T})$

- One can handle the quintic nonlinearity in a similar manner but needs an integration by parts trick (Oh–Pocovnicu '16 for 3-d quintic NLW)

· Stochastic nonlinear beam equation (SNLB)

$$\partial_t^2 u = \Delta^2 u - u^p + \xi \quad \text{on } \mathbb{T}^d$$

$$\Psi(t) = \int_0^t \frac{\sin(t\Delta)}{\Delta} dW(t') \in C_t W_x^{2-\frac{d}{2}-, \infty}$$

Thm 8 (Mosincat-Pocovnicu-Tolomeo-Wang '17) $d=3$

Let $1 < p < 11/3$. Then, SNLB is globally well-posed in $H^2 \times L^2$.

Sec 4 Remarks & comments

① We can also consider SPDEs with a multiplicative noise.

$$(\text{SNLS}_m) \quad i\partial_t u = \Delta u \pm |u|^{p-1} u + u \Phi d\xi$$

Ito or Stratonovich product

Key lemma: $p \geq 2$

$$\mathbb{E}\left[\left\| \int_0^t S(t-t') u \Phi dW \right\|^p_{X_T^{s,\frac{1}{2}-}} \right]$$

$$\lesssim C_{T,\Phi} \mathbb{E}\left(\|u\|^p_{X^{s,\frac{1}{2}-}} \right).$$

$\Leftarrow$ B-D-G inequality

As a consequence, we construct solns in

$$L^2(\Omega\,;\, X_T^{s,\frac{1}{2}-} \cap C_T H^s)$$

i.e. NOT pathwise !!

($\Leftarrow$ uses the "truncation" method to handle a nonlinearity.

Q1: pathwise well-posedness of SNLS with a multiplicative noise.

Q2: pathwise well-posedness of SNLS with a multiplicative space-time white noise

difficult

$$u \quad dW$$

$$u \in C_t^{\frac{1}{2}-} C_x^{\frac{1}{2}-} \qquad C_t^{-\frac{1}{2}-},\ C_x^{-\frac{1}{2}-}$$

$\left(\frac{1}{2}-\right) + \left(-\frac{1}{2}-\right) < 0 \Rightarrow$ cannot make sense of the product.

② singular stochastic dispersive PDEs

2.ⓐ 2-d SNLW with additive space-time white noise

$$\partial_t^2 u = \Delta u \pm u^k + \xi , \qquad x \in \mathbb{T}^2.$$

Write $u = v + \Psi$. where

$$\Psi = \int_0^t \frac{\sin (t-t')|\nabla|}{|\nabla|} \, dW(t')$$

$\Rightarrow$ formally, $\partial_t^2 v = \Delta v \pm (v + \Psi)^k$

<u>BUT</u> in 2-d, we have

$$\Psi \in C_t H_x^{-\varepsilon} \setminus C_t L_x^2$$

$\Rightarrow$ Ψ^k does <u>not</u> make sense and we need to renormalize the nonlinearity.

- <u>Truncated stochastic convolution</u>

$$\Psi_N = P_{\le N} \Psi$$

$$= \sum_{|n| \le N} e_n(x) \int_0^t \frac{\sin (t-t')|n|}{|n|} \, d\beta_n(\tau)$$

For *fixed* $(x,t) \in \mathbb{T}^2 \times \mathbb{R}_+$,

$\Psi_N(x,t) =$ mean 0 Gaussian r.v. with

variance $\sigma_N(t) = \mathbb{E}\left[\Psi_N^2(x,t)\right]$

$\sim t \log N \longrightarrow \infty$ as $N \to \infty$

- For $u_N = v_N + \Psi_N$,

v_N ← smoother

we have $u_N^k = \sum_{l=0}^{k} \binom{k}{l} \Psi_N^l \, v_N^{k-l}$

$\Psi_N^l \downarrow \infty$

Wick ordering: replace Ψ_N^l by

$:\Psi_N^l(x,t): \; = \; H_l(\Psi_N(x,t); \sigma_N(t))$ ptwise

Hermite poly of deg l defined by

$$e^{tx - \frac{1}{2}\sigma t^2} = \sum_{l=0}^{\infty} \frac{t^l}{l!} H_l(x;\sigma).$$

Rmk: Wick ordering of Ψ^l = orthogonal projection onto the Wiener chaoses of deg l

By repeating a computation analogous to Prop 1 on page 6, we have

Prop 9: $\{:\Psi_N^\ell:\}$ is a Cauchy seq in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$

$$\Rightarrow :\Psi^\ell: \stackrel{\text{def}}{=} \lim_{N\to\infty} :\Psi_N^\ell:$$

By $H_k(x+y) = \sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(x)\, y^{k-\ell}$,

we have

$$:u_N^k(x,t): \stackrel{\text{def}}{=} H_k(u_N(x,t); \sigma_N(t))$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(\Psi_N(x,t); \sigma_N(t)) \big(v_N(x,t)\big)^{k-\ell}$$

$$\Big\downarrow \quad N\to\infty$$

$$\boxed{:u^k(x,t): \stackrel{\text{def}}{=} \sum_{\ell=0}^{k} \binom{k}{\ell} :\Psi^\ell(x,t):\big(v(x,t)\big)^{k-\ell}}$$

$$=: F_\Psi(v(t))$$

for $u = v + \Psi$
$\quad\ \uparrow$ smoother

(NOT defined for all functions!!)

$\Leftarrow$ Time-dependent renormalization

$\Rightarrow$ consider the fixed pt problem for v

$$v(t) = S(t)(\phi_0, \phi_1) \mp \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F_{\Psi}(v(t'))\,dt'$$

Thm 10: (Gubinelli-Koch-Oh '17)

Wick ordered SNLW

(WSNLW) $\partial_t^2 u - \Delta u \pm :u^k: = \xi$

is pathwise locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$

(i) $k \geq 4$; $s \geq s_{crit}$

(ii) $k = 2, 3$: $s > s_{crit}$,

where

$s_{crit} = \max(s_{scaling}, s_{conf}, 0)$

scaling critical regularity $= \frac{d}{2} - \frac{2}{k-1}$

conformal critical regularity $= \frac{d+1}{4} - \frac{1}{k-1}$

$$u = \Psi + v$$

$$\in \Psi + C([0,T]; H^{\sigma}(\mathbb{T}^2))$$

$$\subset C([0,T]; H^{-\varepsilon}(\mathbb{T}^2)), \quad \forall \varepsilon > 0$$

$$(\sigma = \min(s, 1-\varepsilon))$$

Idea: Solve the fixed pt problem for v by

① Strichartz estimates,

② Prop 9 $\left(:\Psi^{\ell}: \in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)) \right)$

③ product estimate : $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}$

$$\| \langle \nabla \rangle^{-\varepsilon} (fg) \|_{L^r} \lesssim \| \langle \nabla \rangle^{-\varepsilon} f \|_{L^p} \| \langle \nabla \rangle^{\varepsilon} g \|_{L^q}$$

Global well-posedness?

defocusing cubic WSNLW

$$\partial_t^2 v - \Delta v + v^3 = -3v^2\Psi - 3v :\Psi^2: - :\Psi^3:$$

Two difficulties:

① $v(t) \notin H^1$

$\Leftarrow$ can not use the energy:

$$E(v) = \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{2}\int |\nabla v|^2 + \frac{1}{4}\int v^4$$

We need to smooth v.

$\Rightarrow$ I-method in the stochastic setting.

② Even if v were in H^1, v does not satisfy (deterministic) NLW, i.e. $E(v)$ is not conserved.

$\Rightarrow$ Need a Gronwall-type argument as in pp. 24–25.

Thm 11: (Gubinelli-Koch-Oh-Tolomeo '17)

The defocusing cubic WSNLW is globally well-posed

2.6 Open questions (difficult)

- 1-d SNLS with space-time white noise forcing

$$i \partial_t u = \partial_x^2 u \pm |u|^2 u + \xi$$

need a renormalization

- 3-d SNLW with space-time white noise forcing

$$\xi \in C_t^{-\frac{1}{2}-} H_x^{-\frac{3}{2}-}$$

$$\Rightarrow \Psi \in C_t H_x^{-\frac{1}{2}-},$$